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Quantic Lattices

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The category of quantic lattices is defined. All the multiplicative lattices, such as *residuated lattices* and *orthomodular lattices*, turn out to be objects of this new category.

1. INTRODUCTION

When studying lattices it is not unusual to encounter lattices that naturally possess three binary operations, that is, the usual $\wedge$ (meet) and $\vee$ (join) plus a "product" which we shall denote by &. Perhaps the best-known example of such a gadget is the lattice of two-sided ideals of a ring R.

The study of such lattices is by no means new, since it can be traced back to Krull (1924) and Ward and Dilworth (1939). Their aim was to study the lattice of ideals of a ring R; to that purpose they defined the concept of *residuated lattice.* Almost 50 years later Mulvey (1984) and Borceux (1984) reinitiated the study of complete residuated lattices under the new name of *quantale* (a complete residuated lattice is a—not necessarily idempotent—quantale), assuming it would be related to the logic of quantum mechanics or quantum logic, for short, though its connection to this logic is still very obscure. On the other hand, the study of quantum logic had been going on since 1936, when Birkhoff and von Neumann (1936) wrote their paper "The logic of quantum mechanics." They showed that, given a quantum mechanical system S, the set of propositions about S constitutes what is usually called an *orthomodular lattice* (Kalmbach, 1983). Finch (1970) emphasized the existence of a natural "product" operation in such a lattice. Though they share common properties, quantales and orthomodular lattices are distinctly different, as we shall see.

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In this paper we construct a category containing both orthomodular lattices and quantales as its objects. The name "quantic lattice" is adopted to keep up with the tradition that lattices of this type are usually associated with quantum logic. We give several examples, paying particular attention to the lattice of preradicals on an Abelian category.

2. THE NEW LATTICE

Definition 2.1. The category of *quantic lattices* Q is defined as follows: an object of Q is a complete lattice $(Q, \leq)$ provided with a binary operation $\&: Q \times Q \to Q$ satisfying:

(a) $-\&q: Q \to Q$ is a poset morphism for all q in Q.
(b) $-\&q$ has a right adjoint $q \to -$ for all q in Q.

A morphism of quantic lattices from Q_1 to Q_2 is a map $f: Q_1 \to Q_2$ such that:

(i) $f(1) = 1$.
(ii) $f(\vee_j q_j) = \vee_j f(q_j)$ for any family $\{q_j\}_j \subseteq Q$.
(iii) $f(p\&q) \geq f(p)\&f(q)$ for all p, q in Q.

If equality holds in (iii), then the morphism is said to be strict.

Observe that adjointness just means that $p\&q \leq r$ iff $p \leq q \to r$ for all p, q, and r in Q. It is also clear, by the adjoint functor theorem, that b can be substituted by:

(b′) $(\vee_j q_j)\&p = \vee_j (q_j\&p)$ for all p in Q, $\{q_j\}_j \subseteq Q$.

In any quantic lattice we have that $0\&q = 0$ and $q \to 1 = 1$ for all q in Q.

Examples

1. Any orthomodular lattice L with $x\&y = (x \vee y^{\perp}) \wedge y$, for all x, y in L. Here, $\perp$ denotes orthocomplementation [see Finch (1970) for details].

2. Any *locale* with $a\&b = a \wedge b$ for all a, b in L.

3. Any complete residuated lattice in the sense of Ward and Dilworth (1939), where & is the lattice product.

4. Any quantale (Borceux, 1984); morphisms of quantales are morphisms of quantic lattices.

5. Let Q be any complete lattice and $j: Q \to Q$ a poset morphism such that given $q \in Q$, $\{p_i\}_i \subseteq Q$ we have $(\vee_i p_i) \wedge j(q) = \vee_i (p_i \cap j(q))$. Then $p\&q = p \wedge j(q)$ satisfies the required properties. Note that if Q is a locale, then $j: Q \to Q$ can be any poset morphism. If $j(p \wedge j(q)) = j(p) \wedge j(q)$, then & is associative [see Borceux (1984) for details].

6. Let Q be the locale of open sets of a topological space X; put $j: Q \to Q$;

$$a \mapsto \neg\neg a$$

where $\neg\neg a$ is the interior of the closure of a. By the above, $a\&b = a \wedge (\neg\neg b)$ for all a, b in Q. Note that if $a\&b = b$, then the map $-\&b$: $[0, a] \to [0, b]$ is a morphism of quantic lattices where $[0, a]$ and $[0, b]$ have the induced structure ($[0, a] = \{x \in Q \mid x \leq a\}$).

7. The lattice of two right (left)-sided ideals of a ring R with the usual meet, join, and product.

8. The lattice of *preradicals* on an Abelian category $\underline{A}$. This example will be described in detail in Section 3.

Borrowing the terminology from Niefield and Rosenthal (1987), we say that an element q of a quantic lattice Q is:

(i) Idempotent iff $q\&q = q$.
(ii) Right-sided iff $q1 = q$.
(iii) Left-sided iff $1\&q = q$.
(iv) Two-sided iff (ii) and (iii) hold.

Note that if q is orthomodular with $x\&y = (x \vee y^{\perp}) \wedge y$ for all x, y in Q, then (i) and (iv) are satisfied. Q will be said to be idempotent or right (left, two, resp.)-sided if the corresponding property is satisfied for all q in Q. One says that Q is *associative* iff the & operation is associative. Quantales are associative, but orthomodular lattices are not necessarily so; as a matter of fact, they are associative if they constitute a Boolean algebra, as the next proposition shows.

Proposition 2.2. Let L be an orthomodular lattice; then the operation & is associative iff L is a Boolean algebra.

Proof. Suppose & is associative; then we claim that

$$a \to b^{\perp} = (a\&b)^{\perp}$$

Indeed

$$(a \to b^{\perp})\&(a\&b) = ((a \to b^{\perp})\&a)\&b \leq b^{\perp}\&b = 0$$

Given x in L such that $x\&(a\&b) = 0$, then

$$x\&(a\&b) = 0 \Rightarrow (x\&a)\&b = 0$$

so that $x \leq a \to b^{\perp}$.

Therefore $a \to b^{\perp} = (a\&b)^{\perp}$; from Román and Rumbos (1988) we have that $a \to b^{\perp} = (b\&a)^{\perp}$, yielding $b\&a = a\&b$ for all a, b in L. So that again from Román and Rumbos (1988), L is a Boolean algebra. ∎

The *implication* $q \rightarrow -$ has been studied for orthomodular lattices (e.g., Finch, 1970; Hardegree, 1981; Román and Rumbos, 1988); it is worth mentioning that the concepts of "importation–exportation" and "residuations" used in Hardegree (1981) are just adjoint functor relations.

Lemma 2.3. If Q is a left-sided quantic lattice, then $p \leq q$ iff $p \rightarrow q = 1$ for all p, q in Q. ■

Proof. $p = 1\&p \leq q$ iff $1 \leq p \rightarrow q$ iff $1 = p \rightarrow q$. ■

As suggested in Finch (1970) and Román and Rumbos (1988), the operation "&" can be interpreted as a *noncommutative* logical conjunction. If Q is orthomodular or a quantale, "&" generalizes $\wedge$ in the following sense: $p \wedge q \leq p\&q$ for all p, q in Q. For a quantic lattice we have the next result.

Proposition 2.4. If Q is a left-sided quantic lattice, $p \wedge q \leq p\&q$ iff $p \rightarrow q = p \rightarrow (p\&q)$ for all p, q in Q.

Proof. $(\Rightarrow)$. $p \rightarrow q \geq p \rightarrow (p\&q)$ always hold. On the other hand, $p \rightarrow q \leq p \rightarrow (p\&q)$ iff $(p \rightarrow q)\&p \leq p\&q$, but $(p \rightarrow q)\&p \leq 1\&p \leq p$ and $p \rightarrow q \leq p \rightarrow q$ implies $(p \rightarrow q)\&p \leq q$, so that $(p \rightarrow q)\&p \leq p \wedge q \leq p\&q$ and hence, $p \rightarrow q = p \rightarrow (p\&q)$.

$(\Leftarrow)$. By Lemma 2.3, $1 = p \wedge q \rightarrow p$, so that $p \wedge q \rightarrow p \leq p \leq q \rightarrow (p \wedge q)\&p$ implies $p \wedge q \rightarrow (p \wedge q)\&p = 1$; hence $p \wedge q \leq (p \wedge q)\&p \leq p\&q$, which is the desired result. ■

In the case of quantales it turns out that $p\&-$ is a poset morphism for all p in Q. If we have a two-sided quantic lattice Q satisfying $p \wedge q \leq p\&q$ (for example, Q orthomodular), then this property implies that Q is a *locale*, as the next proposition shows; if Q is orthomodular, then this, of course, implies it is actually a Boolean algebra.

Proposition 2.5. Let Q be any two-sided quantic lattice satisfying $p \wedge q \leq p\&q$ for all p, q in Q; then, if $p\&-$ is a poset morphism for all p in Q, Q is a locale.

Proof. Given p, q in Q, we have $p\&q \leq 1\&q = q$ and $p\&q \leq p\&1 = p$; hence $p\&q \leq p \wedge q$, yielding $p \wedge q = p\&q$. Then Q is a locale since $- \wedge q$ will distribute arbitrary joins for all q in Q. ■

Let Q be a $\vee$-lattice; then End(Q) (the $\vee$-lattice endomorphisms) is a quantic lattice if we define $\leq$, $\wedge$, $\vee$ pointwise in End(Q) and $f\&g = g \circ f$ for all f, g in End(Q). Now take Q a quantic lattice and consider the map $f\colon Q \rightarrow \mathrm{End}(Q)$;

$$f\colon b \mapsto f_b$$

is given by $f_b(a)=a\&b$, $\forall a\in Q$. Clearly f_b belongs to End(Q), since $-\&b$ distributes arbitrary joins for all b in Q. If $a\&-$ has a right adjoint for all a in Q, then $f(\wedge_i b_i)=f_{\vee_i b_i}$ for $\{b_i\}_i\subseteq Q$, so that f is a $\vee$-lattice morphism; furthermore, if Q is associative, then $f_{b\&c}(a)=a\&(b\&c)=(a\&b)\&c=f_c\circ f_b(a)=(f_b\&f_c)(a)$, so that f is actually a strict morphism of quantic lattices. Let $E(Q)$ denote the image of f in End(Q). The following is an interesting result.

Theorem 2.6. Let Q be a quantale in the sense of Borceux (1984); then, the correspondence $\eta: Q\rightarrow E(Q)$ yields a functor from quantales to locales.

Proof. For such a quantale Q, the relation $a\&b\&c=a\&c\&b$ holds for all a, b, c in Q (Borceux, 1984); hence, $f_{b\&c}=f_b\circ f_c=f_c\circ f_b=f_b\wedge f_c$, so that indeed $E(Q)$ is a locale.

By the above observations f is a quantale morphism. Moreover, given any Φ: $Q_1\rightarrow Q_2$ a quantale map, then $E(Q_1)\rightarrow E(Q_2)$,

$$f_b\mapsto f_{\Phi(b)}$$

yields a map of locales, so η is indeed a functor. ■

Observe that $E(Q)$ is a *localic* quotient of Q in the sense of Niefield and Rosenthal (1987).

When Q is an orthomodular lattice, $E(Q)$ is also orthomodular by letting $f_a\wedge f_b=(f_{a^\perp}\circ f_b)'\circ f_b$ and $(f_a)^\perp=f_{a^\perp}$ for all a, b in Q. Furthermore, η is an isomorphism of orthomodular lattices. It turns out that $E(Q)$ is exactly the lattice of closed projections of a certain Baer $*$-semigroup; this result was proved by Foulis (1960).

The construction of *quantic nuclei* and *quantic conuclei* in quantic lattices can be carried out as in Niefield and Rosenthal (1987). It turns out that given a quantic lattice Q and j: $Q\rightarrow Q$ a closure operator, then j is a quantic nucleus iff $j(a\&b)\geq j(a)\&j(b)$ and $j(a\&b)=j(j(a)\&b)$. The second condition is implied by the first one whenever $a\&-$ is a poset morphism for all a in Q. As in Niefield and Rosenthal (1987), the image of j is a quantic quotient lattice of Q. Similarly, if j: $Q\rightarrow Q$ is a coclosure operator, then j is a conucleus iff $j(a\&b)=j(a)\&j(b)$; in this case the image of j is a quantic sublattice of Q. Consider the following example:

Let Q be an orthomodular lattice; $Z(Q)$ will denote the *center* of Q, that is, $Z(Q)=\{a\in Q\mid a\&b=b\&a\ \forall b\in Q\}$.

From Finch (1970) we know that $Z(Q)=\{a\in Q\mid a\&b=a\wedge b\ \forall b\in Q\}$ and also is equal to $\{a\in Q\mid b\&a=a\wedge b\ \forall b\in Q\}$. Now, $Z(Q)$ is trivially closed under arbitrary meets and furthermore we have:

Lemma 2.7. $Z(Q)$ is closed under arbitrary joins.

Proof. Let $\{a_i\}_i \subseteq Z(Q)$, $b \in Q$; then $(\vee_i a_i) \wedge b = (\vee_i a_i)\&b = \vee_i (a_i\&b)$ $\vee_i (a_i \wedge b) \leq (\vee_i a_i) \wedge b$; hence, $(\vee_i a_i) \wedge b$ for all b in Q, yielding $\vee_i a_i \in Z(Q)$. ■

Let's define a coclosure operator $c\colon Q \to Q$ as follows. Given $a \in Q$, $c(a) = \vee \{x \in Z(Q) \mid x \leq a\}$; this union is nonempty since $0 \in Z(Q)$ and $0 \leq a$ for all a in Q.

Proposition 2.8. c is a conucleus in Q; moreover, the image of c is a Boolean subalgebra of Q equal to $Z(Q)$.

Proof. We have to prove that $c(a\&b) \geq c(a)\&c(b)$ for all a, b in Q.

$$c(a)\&c(b) = c(a) \wedge c(b) \in Z(Q)$$

and $c(a) \wedge c(b) \leq a \wedge b$, yielding $c(a) \wedge c(b) \leq c(a \wedge b)$, but also $a \wedge b \leq a\&b$ implies $c(a \wedge b) \leq c(a\&b)$, so that finally $c(a\&b) \geq c(a) \wedge c(b) = c(a)\&c(b)$.

The image of c is clearly $Z(Q)$ and it is a Boolean algebra since $c(a) \wedge c(b) = c(a)\&c(b)$, so that the meet distributes over arbitrary joins. ■

3. THE QUANTIC LATTICE OF PRERADICALS

Let $\underline{A}$ be any Abelian category and $1\colon \underline{A} \to \underline{A}$ the identity functor. As usual, a preradical on $\underline{A}$ is a subfunctor of $1\colon \underline{A} \to \underline{A}$; the preradicals of $\underline{A}$ will be denoted by Prerad $-\underline{A}$. They can be turned into a complete lattice as follows:

Given σ, τ in Prerad $\underline{A}$ and $\{\sigma_i\}_i \subseteq$ Prerad $\underline{A}$, then $\sigma \leq \tau$ iff $\sigma(A) \leq \tau(A)$ for all A in $\underline{A}$. We have

$$(\wedge_i \sigma_i)(A) = \bigcap_i \sigma_i(A)$$

and

$$(\wedge_i \sigma_i)(\mathrm{A}) = \sum_i \sigma_i(\mathrm{A})$$

for all A in $\underline{A}$.

There are at least two ways of turning Prerad $\underline{A}$ into a quantic lattice: we can define $\sigma\&\tau = \sigma \circ \tau$ (composition of preradicals) or $\sigma\&\tau = (\tau : \sigma)$ where $(\tau : \sigma)$ is defined as in Stenström (1985) by

$$\frac{(\tau : \sigma)(A)}{\tau(A)} = \sigma(A/\tau(A))$$

for all A in $\underline{A}$. It can be easily verified that in both cases, $-\&\tau$ is a poset morphism that distributes arbitrary joins and *as a matter of fact*, also over arbitrary meets!, so that the opposite category (Prerad $\underline{A}$)$^{\mathrm{op}}$ is also a quantic lattice. It is easy to see that with either product, Prerad $\underline{A}$ is an associative two-sided quantic lattice. If $\sigma \in$Prerad $\underline{A}$ is such that $\sigma \circ \sigma = \sigma$, then σ is an idempotent preradical and if $(\sigma : \sigma) = \sigma$, then it is a radical.

From now on let $\underline{A} = \underline{R\text{-mod}}$, the category of left $\underline{R}$-modules for some ring R. For simplicity Prerad $\underline{R\text{-mod}}$ will be denoted by Prerad $\underline{R}$. There are several closure and coclosure operators in Prerad $\underline{R}$; here we only take a look at two of them: the left exact or *hereditary* and the cohereditary coclosure. We made this choice because left exact preradicals have recently made a comeback (Golan, 1987).

Given $\sigma \in$Prerad $\underline{R}$, a left exact preradical $h(\sigma)$ is defind by $h(\sigma)(M) = M \wedge \sigma(E(M))$ for all $M \in \underline{R\text{-mod}}$; here $E(M)$ denotes the injective hull of M. In Bican *et al.* (1976) the following facts are proved:

(i) $\sigma \leq h(\sigma)$.
(ii) $h(\sigma) = \{\tau | \sigma \leq \tau$ and τ is left exact$\}$.
(iii) $h(\sigma)$ is left exact; i.e., it is hereditary.
(iv) $h(\sigma) \circ \tau = h(\sigma) \wedge \tau \ \forall \tau \in$Prerad $\underline{R}$.

We have, then, that h is a closure operator on Prerad $\underline{R}$ or a coclosure operator on (Prerad $\underline{R})^{\text{op}}$. The image of h consists of all the left exact preradicals. Golan (1987) calls this set R-fil since they correspond exactly to the topologizing filters of right ideals. R-fil is then closed under arbitrary meets and if $\bigcup$ is the join in R-fil, $\sigma \bigcup \tau = h(\vee \tau)$, where τ, $\sigma \in R$-fil and $\vee$ is the join in Prerad $\underline{R}$.

The product $h(\sigma) \circ h(\tau)$ is just $h(\sigma) \wedge h(\tau)$ in r-fil and $(h(\sigma) : h(\tau))$ coincides with the product of filters defined by Golan (1987). It is not hard to see that $(h(\sigma) : h(\tau))$ is left exact, so that $(h(\sigma) : h(\tau)) \geq h(\sigma : \tau)$; it turns out that h is a conucleus in (Prerad $\underline{R})^{\text{op}}$ and hence, $(R\text{-fil})^{\text{op}}$ is a quantic sublattice of (Prerad $\underline{R})^{\text{op}}$ with the "$(\sigma : \tau)$ product." In particular, if $\{\tau_i\}_i \leq R$-fil and $\sigma \in R$-fil, we have

$$(\sigma : \wedge_i \tau_i) = \wedge_i (\sigma : \tau_i)$$

The "residuation" defined by Golan (1987) is, as expected, just the left adjoint to the functor $(\tau : -)$, $\tau \in R$-fil.

Given any ideal I of R, $\eta(I)$ is the unique preradical satisfying

$$x \in \eta(I)(M) \Leftrightarrow I \leq \{r \in R | xr = 0\}$$

In Golan (1987) the following are proved:

(i) $\eta(IJ) = (\eta(I) : \eta(J))$ for all ideals I, J in R.
(ii) $\eta(\sum_i J_i) = \vee_i \eta(J_i)$ for any family $\{J_i\}_i$ of ideals of R.

Within the context of quantic lattices we have that

$$\eta: \quad \text{Idl}\, R \to (\text{Prerad}\, \underline{R})^{\text{op}}$$

is a strict morphism of quantic lattices, where Idl R is the set of two-sided ideals of R. Clearly η is injective and the image consists exactly of all the Jansian filters (Golan, 1987).

Let us now consider the dual case. As in Bican *et al.* (1976), a coclosure operation in Prerad $\underline{R}$ is defined by ch: Prerad $\underline{R} \to$ Prerad $\underline{R}$, where

$$\tau \mapsto \text{ch}(\tau)$$

$\text{ch}(\tau)(M) = \tau(R)M$ for all $M \in R$-mod. The following are proved in Bican *et al.* (1976):

(i) $\text{ch}(\tau) \leq \tau$.
(ii) $\text{ch}(\tau) = \sum\{\sigma | \sigma \leq \tau \text{ and } \tau \text{ preserves epis}\}$.
(iii) $\text{ch}(\tau)$ preserves epis; i.e., it is cohereditary.
(iv) $(\sigma : \text{ch}(\tau)) = \sigma \vee \text{ch}(\tau)$ for all $\sigma \in \text{Rad}\, \underline{R}$.

It is straightforward to check that the composition $(\text{ch}(\tau)) \circ (\text{ch}(\sigma))$ is cohereditary, so that $(\text{ch}(\tau)) \circ (\text{ch}(\sigma)) \leq \text{ch}(\tau \circ \sigma)$ and ch is a conucleus in the quantic lattice Prerad $\underline{R}$ with & as composition of preradicals. The image of ch, i.e., the set of cohereditary radicals, is a quantic sublattice of Prerad $\underline{R}$.

Consider now the map ρ: Idl $R \to$ Prerad $\underline{R}$,

$$I \mapsto r_I$$

where $r_I(M) = IM \ \forall M \in R$-mod. Trivially ρ is a map of quantic lattices; moreover, it is injective and its image is exactly

$$\text{Coh}\, R = \{\sigma \in \text{Prerad}\, \underline{R} \,|\, \sigma \text{ is cohereditary}\}$$

so that Idl R and Coh R are isomorphic quantic lattices.

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